TOPOLOGICAL CONSTRAINTS ON THE RELAXATION OF COMPLEX MAGNETIC FIELDS

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ABSTRACT

Newly emerging magnetic flux can show a complicated linked or interwoven topology of the magnetic field. The complexity of this linkage or knottedness of magnetic flux is related to the free energy stored in the magnetic field. Magnetic reconnection provides a process to release this energy on the time scale of the dynamics. At the same time it approximately conserves the total magnetic helicity. Therefore the conservation of total magnetic helicity is a crucial constraint for the relaxation of complex magnetic fields. However, the total magnetic helicity is only the first, most elementary, quantity of an infinite series of topological invariants of the magnetic field. All these invariants are strictly conserved in ideal magnetohydrodynamics. As an example a preliminary set of these invariants is derived. The relevance of these higher order invariants for the final state of relaxation under magnetic reconnection and their implications for the release of magnetic energy are discussed.

Key words: magnetic fields; topological invariants; magnetic reconnection.

1. INTRODUCTION

Magnetic fields in the corona often show a non-trivial topology, that is the magnetic field lines are interwoven or knotted. They form, in mathematical terms "knots" and "links". Here "knot" refers to a single field line of non-trivial topology, while "link" is used if there are at least two field lines which can not be separated without cutting of lines. Instead of using isolated field lines we can apply these notions to flux tubes as well. Note that the existence of complicated knots and linkages of magnetic flux is in no way artifical. Consider for instance a closed magnetic flux tube. In magnetostatic equilibrium the surfaces of constant pressure are magnetic surfaces, that is the magnetic field is everywhere tangent to the surface. Depending on the ratio of poloidal to toroidal component the field lines will close after a certain number of windings along the flux tube (n) and around the core of the flux tube (m), or they will ergodically fill the whole flux surface. In the former case the flux surface is called a rational surface, while in the latter case its called an irrational surface according to the terminology in tokamak physics. In generic cases each of the infinite set of these rational surfaces has different ratio n/m and therefore for each rational surface the field lines are knotted in form of a so called torus knot of type (n,m). A simple example of these torus knots is shown in Figure 1a).

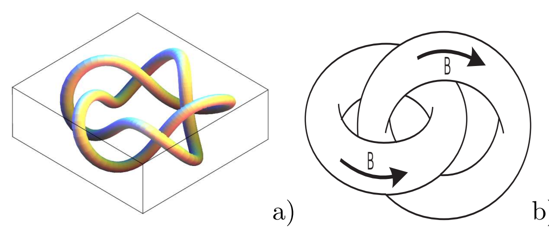

Figure 1. (a) A thin flux tube forming a torus knot of type (n=2,m=5) and (b) a simple linkage of two magnetic flux tubes, which leads to a non-vanishing total helicity and thus to a lower bound of the magnetic energy.

Magnetic fields which show knotted or linked magnetic flux contain a certain amount of free energy. Here the notion "free energy" denotes the difference between the magnetic energy $E(B) = \int_V B^2/(8\pi) d^3r$ of initial configuration and the lowest possible magnetic energy, that is the energy of a vacuum magnetic field satisfying the same boundary conditions. This free energy is stored in the magnetic field and could be set free in a relaxation process. The process of relaxation can be formally separated into two subsequent processes. First an ideal relaxation, that is a relaxation under an ideal Ohm's law, which conserves the magnetic topology and leads to a lower magnetic energy. Second a non-ideal relaxation, for instance by magnetic reconnection, which might finally lead to the vacuum field and thus to the lowest energy state. Of course in reality most likely the reconnection process will set in before the lowest energy state accessible under an ideal evolution is reached. Moreover, it will probably not directly lead to the vacuum field but to a higher energy state, which again might be the starting point of a subsequent relaxation process and might involve further reconnection processes. But nevertheless the difference between the lowest energy state accessible by an ideal evolution and the energy of the corresponding vacuum state is well determined and is refered to as minimum free energy. It is only determined by the topology of the

initial state and boundary conditions.

During a non-ideal relaxation the minimum free energy can be converted in thermal or kinetic energy of the plasma. Thus it might be the source of the energy which is required to heat the solar corona and the determination of the free energy could give an important estimate for the energy available for this process. In the next section we will show the relation between the free energy and the magnetic topology which should motivate the investigation of higher order topological invariants. A preliminary version of such measures of the magnetic topology will be given in Section 3, while the third section is devoted to the evolution of these invariants under reconnection.

2. LOWER BOUNDS FOR THE FREE ENERGY

Roughly speaking the minimum free energy increases with the complexity of the magnetic field. This qualitative statement was given a quantitative meaning in several papers, e.g. Arnold (1986), Freedman (1988), Freedman & He (1991a), Berger (1993), Freedman & He (1991b), showing for instance that the total magnetic helicity,

$$H(\mathbf{B}) = \int_V \mathbf{A} \cdot \mathbf{B} \; d^3x \tag{1}$$

can provide such a lower bound. (For a detailed discussion of magnetic helicity see Brown, Canfield, Pevtsov 1999). In the simplest case, that is if the magnetic field is enclosed in the volume V (no magnetic flux is penetrating the boundary ∂V of the volume , $\mathbf{B} \cdot \mathbf{n}|_{\partial V} = 0$), this reads

$$E(B) \geq C|H(B)|, \tag{2}$$

where C is a constant depending only on the shape and size of the volume V. Hence, if the magnetic field has a non-vanishing total helicity, resulting for instance from a simple linkage of two flux tubes as shown in Figure 1b), the magnetic energy can not decrease below this bound as long as the topology of the field is conserved. A lower energy can only be obtained in an evolution which changes the magnetic topology, like for instance magnetic reconnection. The conservation of topology is usually provided by the ideal evolution of the plasma ($\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$), although the condition is more general and there are several cases of non-ideal evolution which do not change the magnetic topology (Hornig & Schindler 1996).

However, the total helicity is only a very rough measure of the topology of a configuration and the lower bound given in Eq. 2 might be very low or even zero although the topology of the field is non-trivial. An example for a configuration which has a vanishing total magnetic helicity but is non-trivially linked are the so called Borromean rings shown in Figure 2a) and in a topological equivalent configuration in Figure 2b). For this configuration the lower bound given by (2) is zero, but a generalization of such an inequality was given by Freedman & He (1991b) with the help of the so called asymptotic crossing number, which shows that there exists a non-zero lower bound for the energy in this case as well.

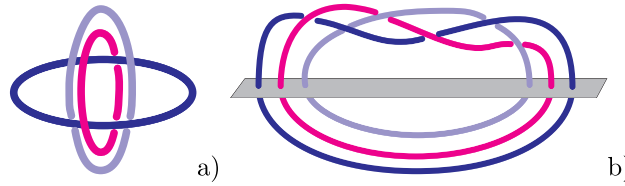

Figure 2. (a) The Borromean rings. A configuration of vanishing total helicity but non-trivially linked. (b) An topological equivalent configuration in form of braided flux tubes.

A drawback of the use of the so called asymptotic crossing number is its complicated and abstract definition. It requires the decomposition of the field in closed flux tubes (which is not possible in general) and a complicated minimization process applied to each combination of linked flux tubes. While this number is important to prove the existence of lower bounds of the energy, it is impossible to calculate it for a generic magnetic field. This is in sharp contrast to the total magnetic helicity, which can easily be calculated for arbitrary magnetic fields irrespective of whether a decomposition into flux tubes exists or not.

This is the motivation to look for invariants, analogous to magnetic helicity, which give non-vanishing values for higher forms of knottedness or linkage of magnetic flux in ideal (topology conserving) evolutions. These higher order topological invariants would provide inequalities similar to Eq. 2. An additional reason for the search for such invariants is that they would lead to a deeper understanding of the global properties of magnetic fields.

3. HIGHER ORDER TOPOLOGICAL INVARIANTS

A formula for higher order topological invariants analogous to the magnetic helicity does not exist yet. However, there are constructions of such invariants if the magnetic field is confined to a set of isolated magnetic flux tubes. Such a formula was given for instance in Berger (1990) for a third order invariant and in Akhmetiev & Ruzmaikin (1995) for a forth order invariant. Note that we are looking for invariants which depend on the magnetic field only, while in the presence of other conserved quantities, such as mass density or entropy, it is possible to combine these and in this way construct new invariants (Tur & Yanovsky 1993).

In the following a method is provide to construct an infinite sequence of higher order invariants which can be calculated for arbitrary magnetic fields, but have the drawback that they require a specific gauge. Therefore, they should be considered as a preliminary form, which, however, allows us to gain insight in the properties of such invariants.

The invariants we are looking for are of the form

$$H^{(n)} = \int_V h^{(n)}(\mathbf{B}) \; d^3r \tag{3}$$

for a magnetic field $\mathbf{B}(x,t)$ enclosed in the (simply connected) volume V. These are topological invariants if for a frozen-in magnetic field,

$$\partial_t \mathbf{B} - \nabla \times (\mathbf{v} \times \mathbf{B}) = 0, \tag{4}$$

the densities $h^{(n)}$ satisfy

$$\partial_t h^{(n)} + \nabla\cdot(\mathbf{v}\ h^{(n)}) = 0. \tag{5}$$

This results for the integral in

$$\frac{d}{dt} H^{(n)} = 0, \tag{6}$$

for a comoving volume V. It is easy to check that Eq. 5 holds for $h^{(n)}$ defined by

$$h^{(n)} := \mathbf{A} \cdot \mathbf{G}^{(n)} \tag{7}$$

provided the vector fields $\mathbf{A}(x,t)$ and $\mathbf{G}(x,t)$ satisfy

$$\partial_t \mathbf{A} + \nabla\,(\mathbf{v}\cdot\mathbf{A}) - \mathbf{v} \times (\nabla \times \mathbf{A}) = 0 \tag{8}$$

$$\partial_t \mathbf{G}^{(n)} - \nabla \times \left(\mathbf{v} \times \mathbf{G}^{(n)}\right) + \mathbf{v}\ \nabla\cdot\mathbf{G}^{(n)} = 0. \tag{9}$$

Now, if $\mathbf{A}$ is the vector potential of $\mathbf{B}$, Eq. 4 yields

$$\partial_t \mathbf{A} - \mathbf{v} \times \nabla \times \mathbf{A} = \nabla \chi. \tag{10}$$

Thus we can meet Eq. 8 with the gauge $\tilde{\mathbf{A}}(x,t) = \mathbf{A}(x,t) + \nabla\Psi(x,t)$ defined by

$$\partial_t \Psi = -\chi - \mathbf{v}\cdot(\mathbf{A} + \nabla\Psi) \tag{11}$$

Note that this equation can be integrated in time for an arbitrary initial gauge $\Psi(x,0)$. Similar, defining $\mathbf{G}^{(n)}$ by

$$\nabla \cdot \mathbf{G}^{(n)} := h^{n-1} \tag{12}$$

Eq. 5 for $n-1$ results in

$$\partial_t \mathbf{G}^{(n)} + \mathbf{v}\ \nabla\cdot\mathbf{G}^{(n)} = \nabla \times \mathbf{F} \tag{13}$$

for some vector field $\mathbf{F}(x,t)$. Again we can meet Eq. 9 using a gauge $\tilde{\mathbf{G}}^{(n)}(x,t) = \mathbf{G}^{(n)}(x,t) + \nabla \times \mathbf{J}(x,t)$ defining $\mathbf{J}$ by

$$\partial_t \mathbf{J} := -\mathbf{F} + \mathbf{v} \times (\mathbf{G}^{(n)} + \nabla \times \mathbf{J}), \tag{14}$$

and the gauge at an initial time is free ($\mathbf{J}(x,0)$), but the equation determines the gauge for all later times.

We can now rename $\tilde{\mathbf{A}}$ by $\mathbf{A}$ and $\tilde{\mathbf{G}}$ by $\mathbf{G}$. Starting with $\mathbf{G}^{(2)} = \mathbf{B}$ equations (7) and (12) yield an infinite recursively defined sequence of integrals of the magnetic field, all of which satisfy Eq. 6. The first ($H^{(2)}$) is the total magnetic helicity, which is a second order invariant in that it depends quadratically on $\mathbf{B}$

$$H^{(2)} = \int_V \mathbf{A} \cdot \mathbf{B}\ d^3 r. \tag{15}$$

The next invariant is of third order

$$H^{(3)} = \int_V \mathbf{A}\cdot\mathbf{G}\ d^3 r \quad \text{with} \quad \nabla\cdot\mathbf{G}^{(n)} := \mathbf{A}\cdot\mathbf{B}, \tag{16}$$

and correspondingly $H^{(n)}$ is of order n. The drawback of this construction is the choice of the special gauge for $\mathbf{A}$ and $\mathbf{G}$, which for instance does not allow to integrate $\mathbf{A}$ by

$$\mathbf{A} = \int_V \nabla_r \frac{1}{|\mathbf{r} - \mathbf{r}'|} \times \mathbf{B}(\mathbf{r}')\ d^3 r', \tag{17}$$

since this corresponds to a different gauge ($\nabla \cdot \mathbf{A} = 0$), which in general does not satisfy Eq. 8. Thus, the value of $H^{(n)}$ is not uniquely determined, but constant for a frozen-in magnetic field.

4. RELAXATION UNDER RECONNECTION

To release the above defined minimum free energy of a magnetic field it is necessary to change its topology. The most important process for astrophysical plasmas which allows for a change of the magnetic topology is magnetic reconnection. Taylor conjectured (Taylor 1974) that the total helicity should be approximately conserved during a relaxation process which involves reconnection. This conjecture proved to be true in that the total helicity is decreasing on a longer time scale than the magnetic energy (Berger 1984). Moreover, there exists a special form of reconnection events which exactly conserve the helicity. This is the case if exactly oppositely directed magnetic fields reconnect such that the magnetic field is locally two dimensional and the electric field is perpendicular to the magnetic field. Then $\mathbf{E}\cdot\mathbf{B} = 0$ and the source term on the right hand side of the balance equation for the helicity density,

$$\frac{\partial \mathbf{A}\cdot\mathbf{B}}{\partial t} + \nabla \cdot (\phi\mathbf{B} + \mathbf{E} \times \mathbf{A}) = -2\ \mathbf{E}\cdot\mathbf{B} \tag{18}$$

vanishes. Integrated over a comoving volume with a surface everywhere tangential to $\mathbf{B}$ and on which the evolution is ideal $\mathbf{E} = -\mathbf{v} \times \mathbf{B}$, Eq. 18 yields

$$\frac{d}{dt} \int_V \mathbf{A}\cdot\mathbf{B}\ d^3 r = -2 \int_V \mathbf{E}\cdot\mathbf{B}\ d^3 r. \tag{19}$$

Thus a reconnection process which satisfies $\mathbf{E}\cdot\mathbf{B} = 0$ exactly conserves the total helicity. But also for the case $\mathbf{E}\cdot\mathbf{B} \neq 0$ the total helicity is usually a well conserved quantity since reconnection processes in a plasma are strongly localized and thus only a small fraction of the volume contributes to the integral on the right hand side of Eq. 19 (Hornig 1999). This means that the lowest energy state accessible under reconnection is not the vacuum field, but a field which has the same value of the total helicity as the initial field. This reduces the minimum free energy for this kind of relaxation and consequently we have to ask whether higher forms of linkage or knottedness will lead to additional constraints and thus to a further reduction of the minimum free energy. For the case of reconnection with $\mathbf{E}\cdot\mathbf{B} = 0$ one can easily prove that this is not the case. The argument is that for this case magnetic reconnection is a two dimensional process and can be represented by a simple cut and paste of magnetic flux (see Freedman & Berger 1993). Using this picture of reconnection one can transform any magnetic field consisting of a set of isolated linked or knotted flux tubes into a set of unlinked and unknotted flux tubes which have at most an internal twist corresponding to the non-vanishing total helicity (see Fig. 3). This confirms the original conjecture

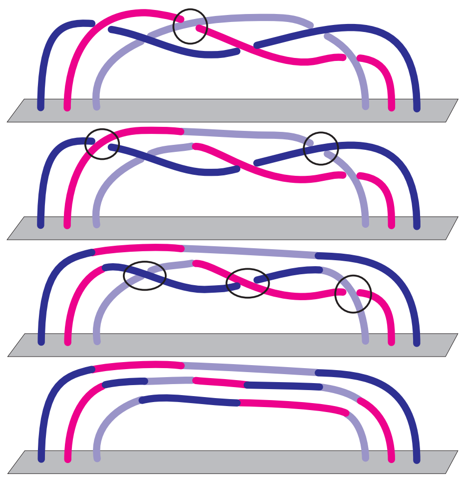

Figure 3. For the example of Fig. 2 a series of six reconnection processes leads to an unlinked field.

of Taylor that the total helicity is the only conserved quantity.

Since the higher order linkage or knottedness is not conserved for the almost ideal magnetic reconnection with $\mathbf{E}\cdot\mathbf{B} = 0$, we do not expect it to be conserved for $\mathbf{E}\cdot\mathbf{B} \neq 0$. However, from the above example of (preliminary) topological invariants it is obvious that those quantities are integrals of high order of the magnetic field. They are global properties of the magnetic field and with each integration, that is with increasing order of the invariants, more and more information of a particular local geometry of the field is lost. On the other hand the magnetic reconnection process is a local process driven by the local geometry of the magnetic field. Thus it is questionable whether the relaxation process is effective in destroying higher forms of knottedness and linkage. With other words, to resolve a complicated link or knot requires a sequence of reconnection processes, and the order and locations of these reconnection processes is not arbitrary (see Fig 3 for an example). Therefore the relaxation might not lead at the right place in the volume to a locally strongly sheared magnetic field, which would be necessary to trigger a reconnection process and resolve the linkage. Moreover, the number of reconnection processes needed to resolve a complicated topology increases with complexity. Thus these invariants, although they may fluctuate under reconnection events, might not decay quickly in generic situations. Note that this does not apply to situations as in most technical plasmas, where the relaxation process or the non-ideal region, respectively, occupies the whole volume under consideration. Only if length scales and the magnetic Reynolds numbers are as high as in astrophysical plasmas magnetic reconnection can be considered as a local process. Also, one has to be cautious in applying these considerations to open field configurations, because in this case all forms of linkage and knottedness have to be defined with respect to a reference field (e.g. the vacuum field).

5. SUMMARY

A complex magnetic field topology can store large amounts of magnetic energy and its minimum magnetic energy under ideal relaxations increases with increasing complexity. Hence such field structures can serve as an efficient energy reservoir. However, the complexity also imposes restrictions on the release of energy from this reservoir. This is because magnetic reconnection is the only process capable to release this energy on a short time scale. But with respect to the first topological invariant, magnetic helicity, reconnection turns out to be inefficient to relax certain configurations to their lowest energy state (the vacuum field), since it approximately conserves this quantity. This is not true for higher order topological invariants, as can be shown easily. However, a complete relaxation in these cases requires a special sequence of reconnection processes, which is unlikely to occur in real plasmas. Thus the higher forms of linkage do not provide a strict constraint for the final state of relaxation by reconnection, as does magnetic helicity, but they will significantly delay the process.

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